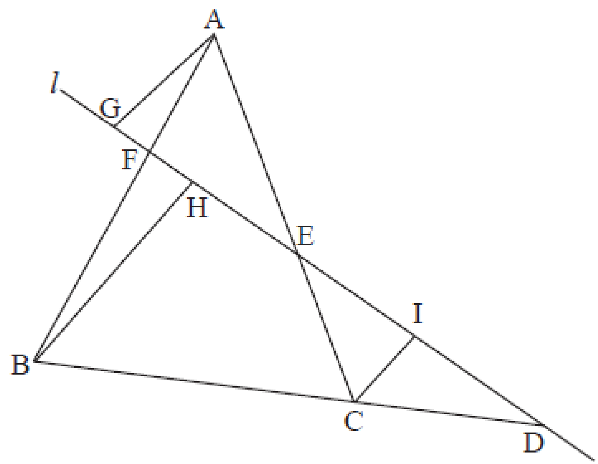
SL Paper 2



a. The diagram shows the line l meeting the sides of the triangle ABC at the points D, E and F. The perpendiculars to l from A, B and C meet l at G, H and I. [13]

(i) State why $\frac{\text{AF}}{\text{FB}} = \frac{\text{AG}}{\text{HB}}$.

(ii) Hence prove Menelaus' theorem for the triangle ABC.

(iii) State and prove the converse of Menelaus' theorem.

b. A straight line meets the sides (PQ), (QR), (RS), (SP) of a quadrilateral PQRS at the points U, V, W, X respectively. Use Menelaus' theorem to show that [7]

$$\frac{\text{PU}}{\text{UQ}} \times \frac{\text{QV}}{\text{VR}} \times \frac{\text{RW}}{\text{WS}} \times \frac{\text{SX}}{\text{XP}} = 1.$$

Markscheme

a. (i) Because the triangles AGF and BHF are similar. ***R1***

(ii) It follows (by cyclic rotation or considering similar triangles) that

$\frac{\text{BD}}{\text{DC}} = \frac{\text{BH}}{\text{IC}}$ ***A1***

and $\frac{\text{CE}}{\text{EA}} = \frac{\text{CI}}{\text{GA}}$ ***A1***

Multiplying these three results gives Menelaus' Theorem, i.e.

$\frac{\text{AF}}{\text{FB}} \times \frac{\text{BD}}{\text{DC}} \times \frac{\text{CE}}{\text{EA}} = \frac{\text{AG}}{\text{HB}} \times \frac{\text{BH}}{\text{IC}} \times \frac{\text{CI}}{\text{GA}}$ ***M1A1***

$= \frac{\text{AG}}{\text{GA}} \times \frac{\text{BH}}{\text{HB}} \times \frac{\text{CI}}{\text{IC}} = -1$ ***M1A1***

(iii) The converse states that if D, E, F are points on the sides (BC), (CA), (AB) of a triangle such that

$\frac{\text{AF}}{\text{FB}} \times \frac{\text{BD}}{\text{DC}} \times \frac{\text{CE}}{\text{EA}} = -1$

then D, E, F are collinear. ***A1***

To prove this result, let D, E, F′ be collinear points on the three sides so that, using the above theorem, ***M1***

$\frac{\text{AF}'}{\text{F}'\text{B}} \times \frac{\text{BD}}{\text{DC}} \times \frac{\text{CE}}{\text{EA}} = -1$ ***A1***

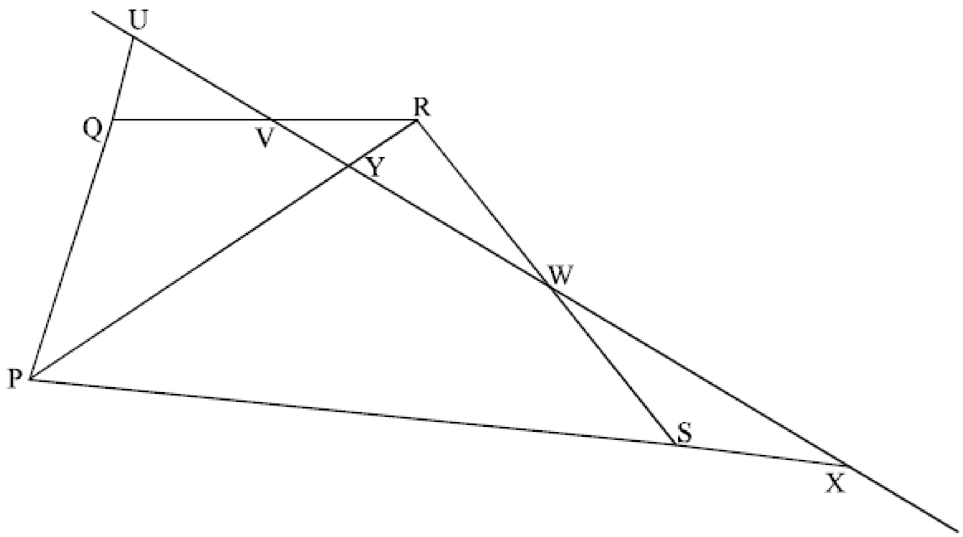
Since $\frac{\text{AF}}{\text{FB}} \times \frac{\text{BD}}{\text{DC}} \times \frac{\text{CE}}{\text{EA}} = -1$ ***M1***

$\frac{\text{AF}'}{\text{F}'\text{B}} = \frac{\text{AF}}{\text{FB}}$ ***A1***

and $\text{F} = \text{F}'$ which proves the converse. ***R1***

[13 marks]

b.



Draw the diagonal PR and let it cut the line at the point Y. ***M1***

Apply Menelaus' Theorem to the triangle PQR. Then,

$\frac{\text{PU}}{\text{UQ}} \times \frac{\text{QV}}{\text{VR}} \times \frac{\text{RY}}{\text{YP}} = -1$ ***M1A1***

Now apply the theorem to triangle PRS.

$\frac{\text{PY}}{\text{YR}} \times \frac{\text{RW}}{\text{WS}} \times \frac{\text{SX}}{\text{XP}} = -1$ ***A1***

$\frac{\text{PU}}{\text{UQ}} \times \frac{\text{QV}}{\text{VR}} \times \frac{\text{RY}}{\text{YP}} \times \frac{\text{PY}}{\text{YR}} \times \frac{\text{RW}}{\text{WS}} \times \frac{\text{SX}}{\text{XP}} = -1 \times -1$ ***M1***

$\Rightarrow \frac{\text{PU}}{\text{UQ}} \times \frac{\text{QV}}{\text{VR}} \times \frac{\text{RW}}{\text{WS}} \times \frac{\text{SX}}{\text{XP}} \times \frac{\text{PY}}{\text{YP}} \times \frac{\text{RY}}{\text{YR}} = 1$ ***A1***

$\Rightarrow \frac{\text{PU}}{\text{UQ}} \times \frac{\text{QV}}{\text{VR}} \times \frac{\text{RW}}{\text{WS}} \times \frac{\text{SX}}{\text{XP}} \times (-1) \times (-1) = 1$ ***(M1)***

$\Rightarrow \frac{\text{PU}}{\text{UQ}} \times \frac{\text{QV}}{\text{VR}} \times \frac{\text{RW}}{\text{WS}} \times \frac{\text{SX}}{\text{XP}} = 1$ ***AG***

[7 marks]

Examiners report

a. [N/A]

b. [N/A]

The circle C has centre O. The point Q is fixed in the plane of the circle and outside the circle. The point P is constrained to move on the circle.

A.a. Show that the opposite angles of a cyclic quadrilateral add up to 180° . [3]

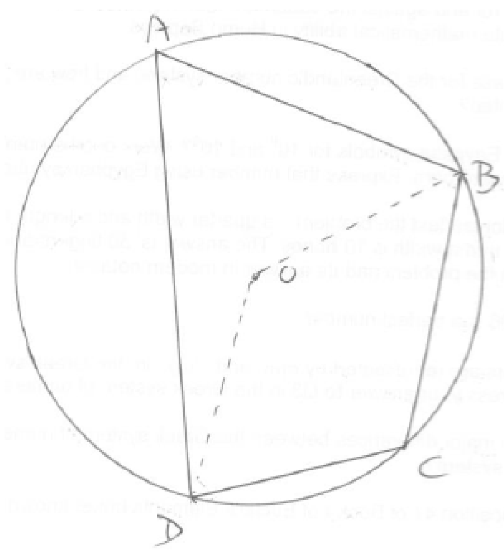
A.b. A quadrilateral ABCD is inscribed in a circle S . The four tangents to S at the vertices A, B, C and D form the edges of a quadrilateral EFGH. Given that EFGH is cyclic, show that AC and BD intersect at right angles. [7]

B.a. Show that the locus of a point P′ , which satisfies $\overrightarrow{\text{QP}'} = k\overrightarrow{\text{QP}}$, is a circle C' , where ***k*** is a constant and $0 < k < 1$. [6]

B.b. Show that the two tangents to C from Q are also tangents to C′ . [4]

Markscheme

A.a.



recognition of relevant theorem ***(M1)***

eg $\text{D}\hat{\text{O}}\text{B} = 2 \times \text{D}\hat{\text{A}}\text{B}$ ***A1***

$360^\circ - \text{D}\hat{\text{O}}\text{B} = 2 \times \text{D}\hat{\text{C}}\text{B}$ ***A1***

so $\text{D}\hat{\text{A}}\text{B} + \text{D}\hat{\text{C}}\text{B} = 180^\circ$ ***AG***

[3 marks]

A.b.

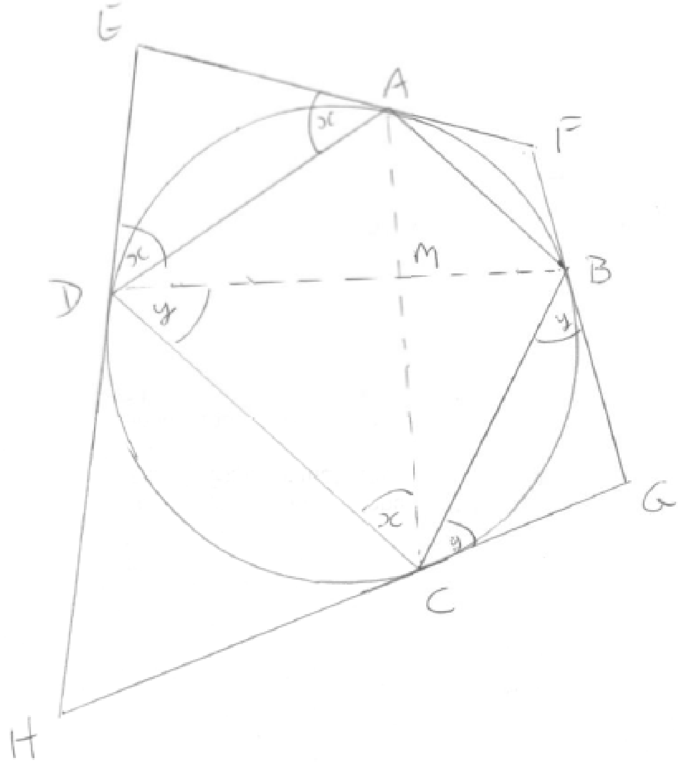


diagram showing tangents EAF, FBG, GCH and HDE; diagonals cross at M. ***M1***

let $x = \text{E}\hat{\text{D}}\text{A} = \text{E}\hat{\text{A}}\text{D}$; $y = \text{B}\hat{\text{C}}\text{G} = \text{C}\hat{\text{B}}\text{G}$ ***A1***

$\text{D}\hat{\text{E}}\text{A} + \text{H}\hat{\text{G}}\text{F} = 180 - 2x + 180 - 2y = 360 - 2(x+y)$ ***M1A1***

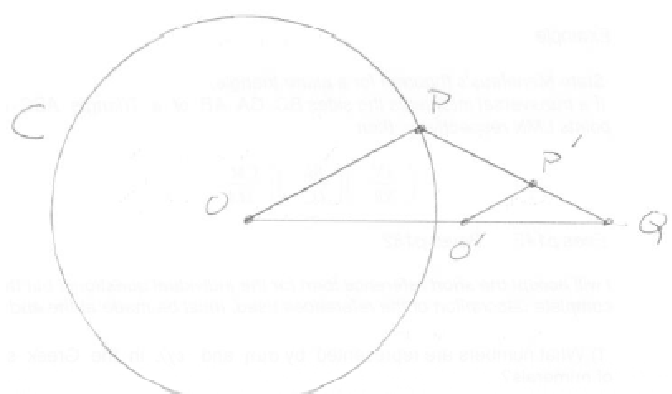
$\text{C}\hat{\text{D}}\text{B} = y$ and $\text{A}\hat{\text{C}}\text{D} = x$, as angles in alternate segments ***M1A1***

$\text{D}\hat{\text{M}}\text{C} = 180 - (x+y) = \left(\frac{1}{2}\right)(\text{D}\hat{\text{E}}\text{A} + \text{H}\hat{\text{G}}\text{F}) = 90°$ ***A1***

so the diagonals cross at right angles ***AG***

[7 marks]

B.a.



M1

let O′ be the point on OQ such O′P′ is parallel to OP ***A1***

using similar triangles $\text{O}'\text{Q} = k\text{OQ}$, so O′ is a fixed point ***M1A1***

and $\text{O}'\text{P} = k\text{OP}$ which is constant ***A1***

so P′ lies on a circle centre O′ ***R1***

so the locus of P′ is a circle ***AG***

[6 marks]

B.b. let one of the two tangents to C from Q touch C at T

the image of T lies on TQ ***A1***

and is a unique point T' on C' ***A1***

so TT' is a common tangent and passes through Q ***R1***

the same is true for the other tangent ***A1***

so the two tangents to C from Q are also tangents to C' ***AG***

[4 marks]

Examiners report

A.a. (a) Most candidates produced a valid answer, although a small minority used a circular argument.

A.b. (b) A few candidates went straight to the core of this question. However, many other candidates produced incoherent answers containing some true statements, some irrelevancies and some incorrect statements, based on a messy diagram.

B.a. (a) This was poorly answered. Many candidates failed to note that the points Q, P and its image were defined to be collinear, and tried to invoke the notion of the Apollonius Circle theory. Others tried a coordinate approach – in principle this could work, but is actually quite tricky without a sensible choice of axes and the origin.

B.b. [N/A]

A circle C passes through the point $(1,\ 2)$ and has the line $3x - y = 5$ as the tangent at the point $(3,\ 4)$.

a. Find the coordinates of the centre of C and its radius. [9]

b. Write down the equation of C. [1]

c. Find the coordinates of the second point on C on the chord through $(1,\ 2)$ parallel to the tangent at $(3,\ 4)$. [5]

Markscheme

a. **METHOD 1**

attempt to exploit the fact that the normal to a tangent passes through the centre $(a,\ b)$ ***(M1)***

EITHER

equation of normal is $y - 4 = -\frac{1}{3}(x - 3)$ ***(A1)***

obtain $a + 3b = 15$ ***A1***

attempt to exploit the fact that a circle has a constant radius: ***(M1)***

obtain $(1 - a)^2 + (2 - b)^2 = (3 - a)^2 + (4 - b)^2$ ***A1***

leading to $a + b = 5$ ***A1***

centre is $(0,\ 5)$ ***(M1)A1***

radius $= \sqrt{1^2 + 3^2} = \sqrt{10}$ ***A1***

OR

gradient of normal $= -\frac{1}{3}$ ***A1***

general point on normal $= (3 - 3\lambda,\ 4 + \lambda)$ ***(M1)A1***

this point is equidistant from $(1,\ 2)$ and $(3,\ 4)$ ***M1***

if $10\lambda^2 = (2 - 3\lambda)^2 + (2 + \lambda)^2$

$10\lambda^2 = 4 - 12\lambda + 9\lambda^2 + 4 + 4\lambda + \lambda^2$ ***A1***

$\lambda = 1$ ***A1***

centre is $(0,\ 5)$ ***A1***

radius $= \sqrt{10\lambda} = \sqrt{10}$ ***A1***

METHOD 2

attempt to substitute two points in the equation of a circle ***(M1)***

$(1 - h)^2 + (2 - k)^2 = r^2,\ (3 - h)^2 + (4 - k)^2 = r^2$ ***A1***

Note: The ***A1*** is for the two LHSs, which may be seen equated.

equate or subtract the equations

obtain $h + k = 5$ or equivalent ***A1***

attempt to differentiate the circle equation implicitly ***(M1)***

obtain $2(x - h) + 2(y - k)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***A1***

Note: Similarly, ***M1A1*** if direct differentiation is used.

substitute $(3,\ 4)$ and gradient $= 3$ to obtain $h + 3k = 15$ ***A1***

obtain centre $= (0,\ 5)$ ***(M1)A1***

radius $= \sqrt{10}$ ***A1***

[9 marks]

b. equation of circle is $x^2 + (y - 5)^2 = 10$ ***A1***

[1 mark]

c. the equation of the chord is $3x - y = 1$ ***A1***

attempt to solve the equation for the chord and that for the circle simultaneously ***(M1)***

for example $x^2 + (3x - 1 - 5)^2 = 10$ ***A1***

coordinates of the second point are $\left(\frac{13}{5},\ \frac{34}{5}\right)$ ***(M1)A1***

[5 marks]

Examiners report

a. This question was usually well done, using a variety of valid approaches.

b. This question was usually well done, using a variety of valid approaches.

c. This question was usually well done, using a variety of valid approaches.

The area of an equilateral triangle is 1 cm^2. Determine the area of:

The points A, B have coordinates (1, 0), (0, 1) respectively. The point P(x, y) moves in such a way that $\text{AP} = k\text{BP}$ where $k \in \mathbb{R}^+$.

A.a. the circumscribed circle. [8]

A.b. the inscribed circle. [3]

B.a. When $k = 1$, show that the locus of P is a straight line. [3]

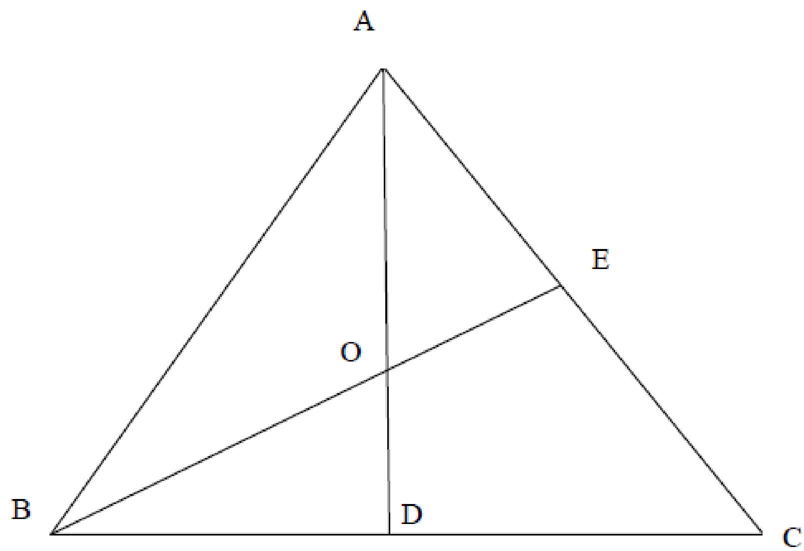
B.b. When $k \neq 1$, the locus of P is a circle. [9]

(i) Find, in terms of k , the coordinates of C, the centre of this circle.

(ii) Find the equation of the locus of C as k varies.

Markscheme

A.a.



consider the above diagram – [AD] and [BE] are the medians and O is therefore both the incentre and the circumcentre ***(R1)***

let $\text{AB} = d$ and let R denote the radius of the circumcircle

then,

$R = \text{AO} = \text{AE}\sec 30^\circ$ ***M1***

$= \frac{d}{2} \times \frac{2}{\sqrt{3}} = \frac{d}{\sqrt{3}}$ ***(A1)***

area of circumcircle $= \pi R^2 = \frac{\pi d^2}{3}$ ***A1***

area of triangle $= \frac{1}{2}\text{AB. AC}\sin\text{BAC}$ ***M1***

$= \frac{\sqrt{3}d^2}{4}$ ***(A1)***

$\frac{\sqrt{3}d^2}{4} = 1 \Rightarrow d^2 = \frac{4}{\sqrt{3}}$ ***A1*** area of circumcircle $= \frac{4\pi}{3\sqrt{3}}$ (2.42) ***A1***

[8 marks]

A.b. let r denote the radius of the incircle

then

$r = \text{OE} = \text{AE}\tan 30^\circ$ ***M1***

$= \frac{d}{2\sqrt{3}}$ ***(A1)***

area of incircle $= \pi r^2 = \frac{\pi d^2}{12}$

$= \frac{\pi}{3\sqrt{3}}$ (0.605) ***A1***

[3 marks]

B.a. $\text{AP}^2 = (x-1)^2 + y^2$ and $\text{BP}^2 = x^2 + (y-1)^2$ ***A1***

$x^2 - 2x + 1 + y^2 = x^2 + y^2 - 2y + 1$ ***M1***

$y = x$ which is the equation of a straight line ***A1***

[3 marks]

B.b. (i) $x^2 - 2x + 1 + y^2 = k^2(x^2 + y^2 - 2y + 1)$ ***M1***

$(k^2-1)x^2 + (k^2-1)y^2 + 2x - 2k^2y + k^2 - 1 = 0$ ***A1***

$x^2 + y^2 + \frac{2x}{k^2-1} - \frac{2k^2y}{k^2-1} + 1 = 0$ ***A1***

by completing the squares or quoting the standard result, ***M1***

coordinates of C are

$\left(-\frac{1}{k^2-1}, \frac{k^2}{k^2-1}\right)$ ***A1***

(ii) let (x, y) be the coordinates of C

attempting to find k or k^2, ***(M1)***

$k^2 = 1 - \frac{1}{x}$ ***(A1)***

$y = \frac{1-\frac{1}{x}}{-\frac{1}{x}}$ ***(M1)***

$y = 1 - x$ ***A1***

[9 marks]

Examiners report

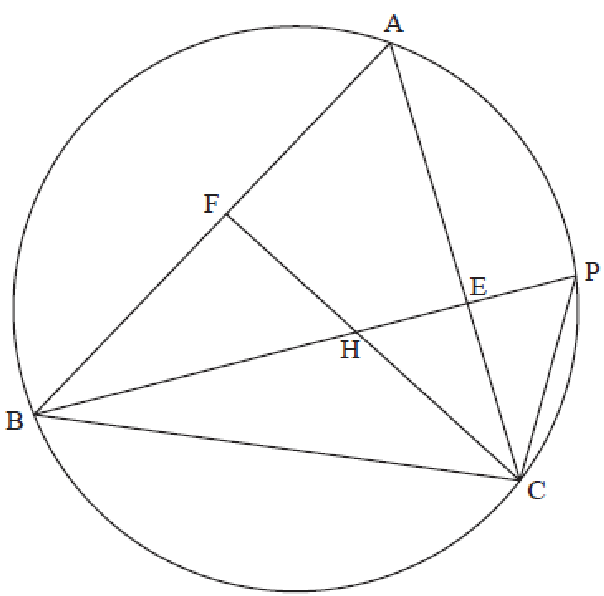
A.a. Most candidates attempted (a) with many different methods seen although some candidates made algebraic errors in applying the appropriate trigonometrical formulae.

A.b. Very few candidates realised that, because the centre of the triangle divides an altitude in the ratio 2 : 1, the area of the inscribed circle is one quarter the area of the circumscribed circle.

B.a. Most candidates solved (a) correctly.

B.b Solutions to (b) were often disappointing. Those candidates who used coordinate geometry often made algebraic errors in obtaining the equation of the circle and then finding the coordinates of its centre. Some candidates tried to use Apollonius' theorem, using the fact that the centre divides the line AB in a ratio dependent upon k, but this approach almost invariably led to algebraic errors.

In the acute angled triangle ABC, the points E, F lie on [AC], [AB] respectively such that [BE] is perpendicular to [AC] and [CF] is perpendicular to [AB]. The lines (BE) and (CF) meet at H. The line (BE) meets the circumcircle of the triangle ABC at P. This is shown in the following diagram.



a. (i) Show that CEFB is a cyclic quadrilateral. [7]

(ii) Show that $\text{HE} = \text{EP}$.

b. The line (AH) meets [BC] at D. [8]

(i) By considering cyclic quadrilaterals show that $\text{C}\widehat{\text{A}}\text{D} = \text{E}\widehat{\text{F}}\text{H} = \text{E}\widehat{\text{B}}\text{C}$.

(ii) Hence show that [AD] is perpendicular to [BC].

Markscheme

a. (i) CEFB is cyclic because $\text{B}\widehat{\text{E}}\text{C} = \text{B}\widehat{\text{F}}\text{C} = 90^\circ$ ***R1***

([BC] is actually the diameter)

(ii) consider the triangles CHE, CPE ***M1***

[CE] is common ***A1***

$\text{H}\widehat{\text{E}}\text{C} = \text{P}\widehat{\text{E}}\text{C} = 90^\circ$ ***A1***

$\text{P}\widehat{\text{C}}\text{E} = \text{P}\widehat{\text{B}}\text{A}$ (subtended by chord [AP]) ***A1***

$P\widehat{B}A = F\widehat{C}E$ (subtended by chord [FE]) ***A1***

triangles CHE and CPE are congruent ***A1***

therefore $HE = EP$ ***AG***

[7 marks]

b. (i) EAFH is a cyclic quad because $A\widehat{E}B = C\widehat{F}A = 90^\circ$ ***M1***

$C\widehat{A}D = E\widehat{F}H$ subtended by the chord [HE] ***R1AG***

CEFB is a cyclic quad from part (a) ***M1***

$E\widehat{F}H = E\widehat{B}C$ subtended by the chord [EC] ***R1AG***

(ii) $A\widehat{D}C = 180^\circ - C\widehat{A}D - D\widehat{C}A$ ***M1***

$= 180^\circ - C\widehat{A}D - (90 - E\widehat{B}C)$ ***A1***

$= 90^\circ - C\widehat{A}D + E\widehat{B}C$ ***A1***

$= 90^\circ$ ***A1***

hence [AD] is perpendicular to [BC] ***AG***

[8 marks]

Examiners report

a. [N/A]
b. [N/A]

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a. Given that the elements of a 2×2 symmetric matrix are real, show that [11]

(i) the eigenvalues are real;

(ii) the eigenvectors are orthogonal if the eigenvalues are distinct.

b. The matrix $\boldsymbol{A}$ is given by [7]

$$\boldsymbol{A} = \begin{pmatrix} 11 & \sqrt{3} \\ \sqrt{3} & 9 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of $\boldsymbol{A}$.

c. The ellipse E has equation $\boldsymbol{X}^T\boldsymbol{A}\boldsymbol{X} = 24$ where $\boldsymbol{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\boldsymbol{A}$ is as defined in part (b). [7]

(i) Show that E can be rotated about the origin onto the ellipse E' having equation $2x^2 + 3y^2 = 6$.

(ii) Find the acute angle through which E has to be rotated to coincide with E' .

Markscheme

a. (i) let $\boldsymbol{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ ***(M1)*** the eigenvalues satisfy

$\det(\boldsymbol{M}-\lambda\boldsymbol{I})=0$ ***(M1)***

$(a-\lambda)(c-\lambda)-b^2=0$ ***(A1)***

$\lambda^2-\lambda(a+c)+ac-b^2=0$ ***A1***

discriminant $=(a+c)^2-4(ac-b^2)$ ***M1***

$=(a-c)^2+4b^2\geq 0$ ***A1***

this shows that the eigenvalues are real ***AG***

(ii) let the distinct eigenvalues be λ_1, λ_2 , with eigenvectors $\boldsymbol{X}_1, \boldsymbol{X}_2$

then

$\lambda_1\boldsymbol{X}_1=\boldsymbol{M}\boldsymbol{X}_1$ and $\lambda_2\boldsymbol{X}_2=\boldsymbol{M}\boldsymbol{X}_1$ ***M1***

transpose the first equation and postmultiply by $\boldsymbol{X}_2$ to give

$\lambda_1\boldsymbol{X}_1^T\boldsymbol{X}_2=\boldsymbol{X}_1^T\boldsymbol{M}\boldsymbol{X}_2$ ***A1***

premultiply the second equation by $\boldsymbol{X}_1^T$

$\lambda_2\boldsymbol{X}_1^T\boldsymbol{X}_2=\boldsymbol{X}_1^T\boldsymbol{M}\boldsymbol{X}_2$ ***A1***

it follows that

$(\lambda 1-\lambda_2)\boldsymbol{X}_1^T\boldsymbol{X}_2=0$ ***A1***

since $\lambda 1\neq\lambda_2$, it follows that $\boldsymbol{X}_1^T\boldsymbol{X}_2=0$ so that the eigenvectors are orthogonal ***R1***

[11 marks]

b. the eigenvalues satisfy $\begin{vmatrix} 11-\lambda & \sqrt{3} \\ \sqrt{3} & 9-\lambda \end{vmatrix}=0$ ***M1A1***

$\lambda^2-20\lambda+96=0$ ***A1***

$\lambda=8, 12$ ***A1***

first eigenvector satisfies

$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ***M1***

$\begin{pmatrix} x \\ y \end{pmatrix}=$ (any multiple of) $\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$ ***A1***

second eigenvector satisfies

$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}=$ (any multiple of) $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$ ***A1***

[7 marks]

c. (i) consider the rotation in which (x, y) is transformed onto (x', y') defined by

$\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ so that $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$ ***M1A1***

the ellipse E becomes

$$\begin{pmatrix} x' & y' \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 11 & \sqrt{3} \\ \sqrt{3} & 9 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 24 \quad \textbf{\textit{M1A1}}$$

$$\begin{pmatrix} x' & y' \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & 12 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 24 \quad \textbf{\textit{A1}}$$

$2(x')^2 + 3(y')^2 = 6$ ***AG***

(ii) the angle of rotation is given by $\cos\theta = \frac{1}{2}, \sin\theta = \frac{\sqrt{3}}{2}$ ***M1***

since a rotational matrix has the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

so $\theta = 60^\circ$ (anticlockwise) ***A1***

[7 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Figure 1

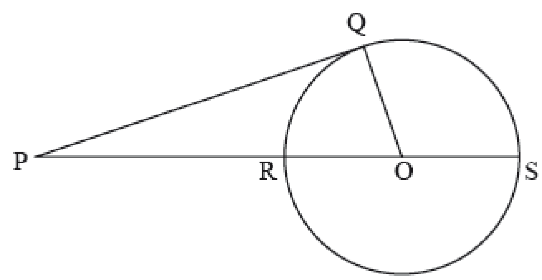

Figure 1 shows a tangent [PQ] at the point Q of a circle and a line [PS] meeting the circle at the points R , S and passing through the centre O of the circle.

Figure 2

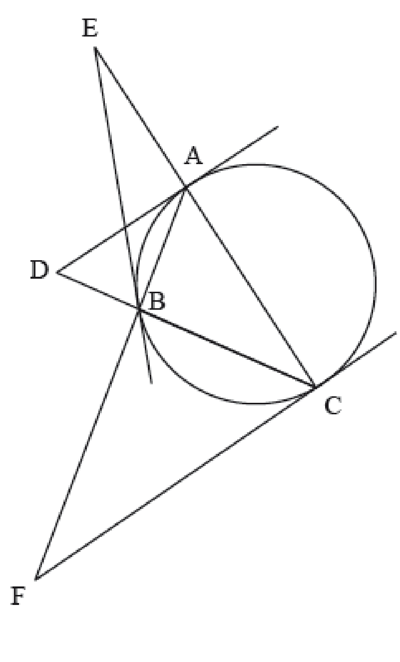

Figure 2 shows a triangle ABC inscribed in a circle. The tangents at the points A , B , C meet the opposite sides of the triangle externally at the points D , E , F respectively.

a.i. Show that $\mathrm{PQ}^2 = \mathrm{PR} \times \mathrm{PS}$. [2]

a.ii. State briefly how this result can be generalized to give the tangent-secant theorem. [2]

b.i. Show that $\frac{\mathrm{AD}^2}{\mathrm{BD}^2} = \frac{\mathrm{CD}}{\mathrm{BD}}$. [2]

b.ii. By considering a pair of similar triangles, show that [2]

$$\frac{\mathrm{AD}}{\mathrm{BD}} = \frac{\mathrm{AC}}{\mathrm{AB}} \text{ and hence that } \frac{\mathrm{CD}}{\mathrm{BD}} = \frac{\mathrm{AC}^2}{\mathrm{AB}^2}.$$

b.iii. By writing down and using two further similar expressions, show that the points D, E, F are collinear. [6]

Markscheme

a.i. let $r =$ radius of circle. Consider

$\mathrm{PR} \times \mathrm{PS} = (\mathrm{PO} - r)(\mathrm{PO} + r)$ ***M1***

$= \mathrm{PO}^2 - \mathrm{OQ}^2$ ***A1***

$= \mathrm{PQ}^2$ because POQ is a right angled triangle ***R1***

[2 marks]

a.ii. the result is true even if PS does not pass through O ***A1***

[2 marks]

b.i. using the tangent-secant theorem, ***M1***

$\mathrm{AD}^2 = \mathrm{BD} \times \mathrm{CD}$ ***A1***

so $\frac{\mathrm{AD}^2}{\mathrm{BD}^2} = \frac{\mathrm{CD}}{\mathrm{BD}} \ldots (1)$ ***AG***

[??? marks]

b.ii. consider the triangles CAD and ABD. They are similar because ***M1***

$\mathrm{D\hat{A}B} = \mathrm{A\hat{C}D}$, angle $\hat{\mathrm{D}}$ is common therefore the third angles must be equal ***A1***

Note: Beware of the assumption that AC is a diameter of the circle.

therefore

$\frac{\mathrm{AD}}{\mathrm{BD}} = \frac{\mathrm{AC}}{\mathrm{AB}} \ldots (2)$ ***AG***

it follows from (1) and (2) that

$\frac{\mathrm{CD}}{\mathrm{BD}} = \frac{\mathrm{AC}^2}{\mathrm{AB}^2}$ ***AG***

[??? marks]

b.iii. two similar expressions are

$\frac{\mathrm{AE}}{\mathrm{CE}} = \frac{\mathrm{BA}^2}{\mathrm{BC}^2}$ ***M1A1***

$\frac{BF}{AF} = \frac{CB^2}{CA^2}$ ***A1***

multiplying the three expressions,

$\frac{CD}{BD} \times \frac{AE}{CE} \times \frac{BF}{AF} = \frac{AC^2}{AB^2} \times \frac{BA^2}{BC^2} \times \frac{CB^2}{CA^2}$ ***M1***

$\frac{CD}{BD} \times \frac{AE}{CE} \times \frac{BF}{AF} = 1$ ***A1***

it follows from the converse of Menelaus' theorem (ignoring signs) ***R1***

that D, E, F are collinear ***AG***

[??? marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b.i. [N/A]
b.ii. [N/A]
b.iii. [N/A]

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The area enclosed by the ellipse is 8π and $b = 2$.

a. Show that the area enclosed by the ellipse is πab. [9]

b.i. Determine which coordinate axis the major axis of the ellipse lies along. [2]

b.ii. Hence find the eccentricity. [2]

b.iii. Find the coordinates of the foci. [2]

b.iv. Find the equations of the directrices. [2]

c. The centre of another ellipse is now given as the point (2, 1). The minor and major axes are of lengths 3 and 5 and are parallel to the x and y axes respectively. Find the equation of the ellipse. [3]

Markscheme

a. $A = 4\int y\mathrm{d}x$ ***(M1)***

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow$

$y = \frac{b\sqrt{a^2-x^2}}{a}$ ***(A1)***

let $x = a\cos\theta \Rightarrow y = b\sin\theta$ ***M1***

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -a\sin\theta$ ***A1***

when $x = 0,\ \theta = \frac{\pi}{2}$. When $x = a,\ \theta = 0$ ***A1***

$\Rightarrow A = 4\int_{\frac{\pi}{2}}^{0} b\sin\theta\,(-a\sin\theta)\,\mathrm{d}\theta$ ***M1***

$\Rightarrow A = -4ab\int_{\frac{\pi}{2}}^{0} \sin^2\theta\,\mathrm{d}\theta$

$\Rightarrow A = -2ab\int_{\frac{\pi}{2}}^{0} (1 - \cos 2\theta)\,\mathrm{d}\theta$ ***M1***

$\Rightarrow A = -2ab\left[\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{0}$ ***A1***

$\Rightarrow A = -2ab\left[0 - 0 - \left(\frac{\pi}{2} - 0\right)\right]$ ***M1***

$\Rightarrow A = \pi ab$ ***AG***

[9 marks]

b.i. $b = 2$

hence $2\pi a = 8\pi \Rightarrow a = 4$ ***A1***

hence major axis lies along the *x*-axis ***A1***

[2 marks]

b.ii $b^2 = a^2\left(1 - e^2\right)$ ***(M1)***

$4 = 16\left(1 - e^2\right) \Rightarrow e = \frac{\sqrt{3}}{2}$ ***A1***

[2 marks]

b.iii coordinates of foci are $(\pm ae,\ 0) = \left(2\sqrt{3},\ 0\right),\ \left(-2\sqrt{3},\ 0\right)$ ***A1A1***

[2 marks]

b.iv equations of directrices are $x = \pm\frac{a}{e} = \frac{8}{\sqrt{3}},\ -\frac{8}{\sqrt{3}}$ ***A1A1***

[2 marks]

c. $a = \frac{3}{2},\ b = \frac{5}{2}$ ***(A1)***

hence equation is $\frac{4}{9}(x-2)^2 + \frac{4}{25}(y-1)^2 = 1$ ***M1A1***

[3 marks]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]
b.iii. [N/A]
b.iv. [N/A]
c. [N/A]

Consider the ellipse having equation $x^2 + 3y^2 = 2$.

a. (i) Find the equation of the tangent to the ellipse at the point $\left(1,\ \frac{1}{\sqrt{3}}\right)$. [7]

(ii) Find the equation of the normal to the ellipse at the point $\left(1,\ \frac{1}{\sqrt{3}}\right)$.

b. Given that the tangent crosses the x-axis at P and the normal crosses the y-axis at Q, find the equation of (PQ). [4]

c. Hence show that (PQ) touches the ellipse. [4]

d. State the coordinates of the point where (PQ) touches the ellipse. [1]

e. Find the coordinates of the foci of the ellipse. [4]

f. Find the equations of the directrices of the ellipse. [1]

Markscheme

a. (i) $2x + 6y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{3y}$ ***A1***

gradient of tangent is $-\frac{\sqrt{3}}{3}$ ***A1***

equation of tangent is $y - \frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}(x-1)$ ***M1A1***

$\left(\Rightarrow y = -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3} + \frac{1}{\sqrt{3}} \Rightarrow y = -\frac{\sqrt{3}}{3}x + \frac{2}{\sqrt{3}}\right)$

(ii) gradient of normal is $\sqrt{3}$ ***A1***

equation of normal is $y - \frac{1}{\sqrt{3}} = \sqrt{3}(x-1)$ ***A1***

$\left(\Rightarrow y = x\sqrt{3} - \sqrt{3} + \frac{1}{\sqrt{3}} \Rightarrow y = \sqrt{3}x - \frac{2}{\sqrt{3}}\right)$

b. coordinates of P are $(2,\ 0)$ ***A1***

coordinates of Q are $\left(0,\ -\frac{2}{\sqrt{3}}\right)$ ***A1***

equation of (PQ) is $\frac{y-0}{x-2} = \frac{\frac{2}{\sqrt{3}}}{2}$ ***M1***

$\Rightarrow y = \frac{1}{\sqrt{3}}(x-2)$ ***A1***

c. substitute equation of (PQ) into equation of ellipse

$x^2 + 3\left(\frac{x-2}{\sqrt{3}}\right)^2 = 2$ ***M1A1***

$\Rightarrow x^2 + x^2 - 4x + 4 = 2$

$\Rightarrow (x-1)^2 = 0$ ***A1***

since the equation has two equal roots (PQ) touches the ellipse ***R1***

d. $\left(1,\ -\frac{1}{\sqrt{3}}\right)$ ***A1***

e. $x^2 + 3y^2 = 2$

$\frac{x^2}{2} + \frac{y^2}{\frac{2}{3}} = 1$

$\Rightarrow a = \sqrt{2},\ b = \sqrt{\frac{2}{3}}$ ***A1***

EITHER

$b^2 = a^2(1 - e^2)$

$\frac{2}{3} = 2(1 - e^2)$ ***M1***

$\Rightarrow e = \sqrt{\frac{2}{3}}$ ***A1***

coordinates of foci are $(\pm ae,\ 0) \Rightarrow \left(\frac{2}{\sqrt{3}},\ 0\right),\ \left(-\frac{2}{\sqrt{3}},\ 0\right)$ ***A1***

OR

$f^2 = a^2 - b^2$ ***M1***

$f^2 = 4 - \frac{2}{3}$ ***A1***

coordinates of foci are $\left(\frac{2}{\sqrt{3}},\ 0\right),\ \left(-\frac{2}{\sqrt{3}},\ 0\right)$ ***A1***

Note: Award accuracy marks if a^2, b^2 and e^2 are given.

f. **EITHER**

equations of directrices are $x = \pm\frac{a}{e} \Rightarrow x = \sqrt{3},\ x = -\sqrt{3}$ ***A1***

OR

$d = \frac{a^2}{f} \Rightarrow x = \sqrt{3},\ x = -\sqrt{3}$ ***A1***

Examiners report

a. Parts a) and b) were well done by most candidates, but surprisingly many candidates lost marks on part c). Parts e) and f) were only completed successfully by a small number of candidates and it was common to see parts a) and b) fully correct, parts c) and d) attempted but not fully correct and parts e) and f) not attempted at all.

b. Parts a) and b) were well done by most candidates, but surprisingly many candidates lost marks on part c). Parts e) and f) were only completed successfully by a small number of candidates and it was common to see parts a) and b) fully correct, parts c) and d) attempted but not fully correct and parts e) and f) not attempted at all.

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d. Parts a) and b) were well done by most candidates, but surprisingly many candidates lost marks on part c). Parts e) and f) were only completed successfully by a small number of candidates and it was common to see parts a) and b) fully correct, parts c) and d) attempted but not fully correct and parts e) and f) not attempted at all.

e. Parts a) and b) were well done by most candidates, but surprisingly many candidates lost marks on part c). Parts e) and f) were only completed successfully by a small number of candidates and it was common to see parts a) and b) fully correct, parts c) and d) attempted but not fully correct and parts e) and f) not attempted at all.

f. Parts a) and b) were well done by most candidates, but surprisingly many candidates lost marks on part c). Parts e) and f) were only completed successfully by a small number of candidates and it was common to see parts a) and b) fully correct, parts c) and d) attempted but not fully correct and parts e) and f) not attempted at all.

The hyperbola with equation $x^2 - 4xy - 2y^2 = 3$ is rotated through an acute anticlockwise angle α about the origin.

a. The point $(x,\ y)$ is rotated through an anticlockwise angle α about the origin to become the point $(X,\ Y)$. Assume that the rotation can be represented by [3]

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Show, by considering the images of the points $(1,\ 0)$ and $(0,\ 1)$ under this rotation that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

b.i. By expressing $(x,\ y)$ in terms of $(X,\ Y)$, determine the equation of the rotated hyperbola in terms of X and Y. [3]

b.ii. Verify that the coefficient of XY in the equation is zero when $\tan\alpha = \frac{1}{2}$. [3]

b.iii. Determine the equation of the rotated hyperbola in this case, giving your answer in the form $\frac{X^2}{A^2} - \frac{Y^2}{B^2} = 1$. [3]

b.iv. Hence find the coordinates of the foci of the hyperbola prior to rotation. [5]

Markscheme

a. consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$ ***(M1)***

the image of $(1,\ 0)$ is $(\cos\alpha,\ \sin\alpha)$ ***A1***

therefore $a = \cos\alpha,\ c = \sin\alpha$ ***AG***

consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$

the image of $(0,\ 1)$ is $(-\sin\alpha,\ \cos\alpha)$ ***A1***

therefore $b = -\sin\alpha,\ d = \cos\alpha$ ***AG***

[3 marks]

b.i. $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$

or $x = X\cos\alpha + Y\sin\alpha,\ y = -X\sin\alpha + Y\cos\alpha$ ***A1***

substituting in the equation of the hyperbola, ***M1***

$(X\cos\alpha + Y\sin\alpha)^2 - 4(X\cos\alpha + Y\sin\alpha)(-X\sin\alpha + Y\cos\alpha)$

$-2(-X\sin\alpha + Y\cos\alpha)^2 = 3$ ***A1***

$X^2(\cos^2\alpha - 2\sin^2\alpha + 4\sin\alpha\cos\alpha) +$

$XY(2\sin\alpha\cos\alpha - 4\cos^2\alpha + 4\sin^2\alpha + 4\sin\alpha\cos\alpha) +$

$Y^2(\sin^2\alpha - 2\cos^2\alpha - 4\sin\alpha\cos\alpha) = 3$

[??? marks]

b.ii. when $\tan\alpha = \frac{1}{2},\ \sin\alpha = \frac{1}{\sqrt{5}}$ and $\cos\alpha = \frac{2}{\sqrt{5}}$ ***A1***

the XY term $= 6\sin\alpha\cos\alpha - 4\cos^2\alpha + 4\sin^2\alpha$ ***M1***

$= 6 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}} - 4 \times \frac{4}{5} + 4 \times \frac{1}{5}\left(\frac{12}{5} - \frac{16}{5} + \frac{4}{5}\right)$ ***A1***

$= 0$ ***AG***

[??? marks]

b.iii. the equation of the rotated hyperbola is

$2X^2 - 3Y^2 = 3$ ***M1A1***

$\frac{X^2}{\left(\sqrt{\frac{3}{2}}\right)^2} - \frac{Y^2}{(1)^2} = 1$ ***A1***

$\left(\text{accept } \frac{X^2}{\frac{3}{2}} - \frac{Y^2}{1} = 1\right)$

[??? marks]

b.iv. the coordinates of the foci of the rotated hyperbola

are $\left(\pm\sqrt{\frac{3}{2}+1},\ 0\right) = \left(\pm\sqrt{\frac{5}{2}},\ 0\right)$ ***M1A1***

the coordinates of the foci prior to rotation were given by

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \pm\sqrt{\frac{5}{2}} \\ 0 \end{bmatrix}$$

M1A1

$\begin{bmatrix} \pm\sqrt{2} \\ \mp\frac{1}{\sqrt{2}} \end{bmatrix}$ ***A1***

[??? marks]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]
b.iii. [N/A]
b.iv. [N/A]

The points D, E, F lie on the sides [BC], [CA], [AB] of the triangle ABC and [AD], [BE], [CF] intersect at the point G. You are given that $\text{CD} = 2\,\text{BD}$ and $\text{AG} = 2\text{GD}$.

A.a. By considering (BE) as a transversal to the triangle ACD, show that [2]

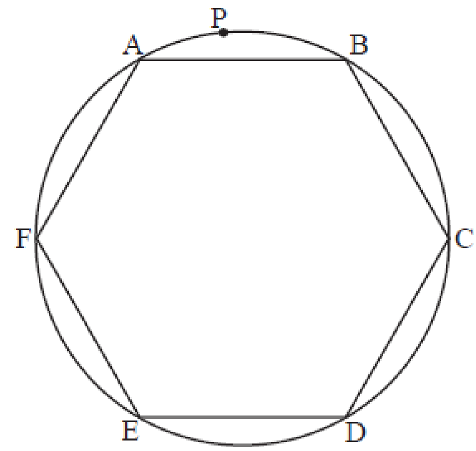
$$\frac{\text{CE}}{\text{EA}} = \frac{3}{2}\ .$$

A.b. Determine the ratios [7]

(i) $\frac{AF}{FB}$;

(ii) $\frac{BG}{GE}$.

B. [7]

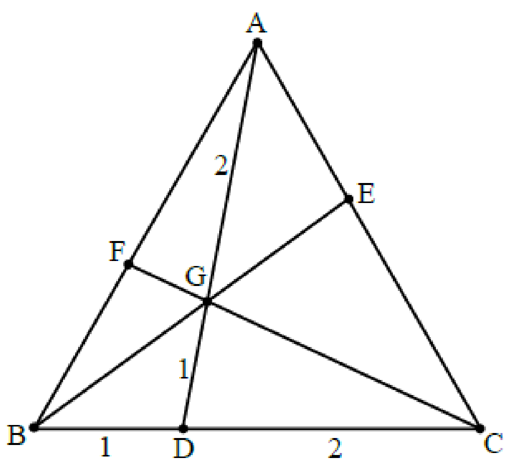


The diagram shows a hexagon ABCDEF inscribed in a circle. All the sides of the hexagon are equal in length. The point P lies on the minor arc AB of the circle. Using Ptolemy's theorem, show that

$$PE + PD = PA + PB + PC + PF$$

Markscheme

A.a.



using Menelaus' theorem in ΔACD ,

$\frac{CE}{EA} \bullet \frac{AG}{GD} \bullet \frac{DB}{BC} = -1$ ***M1***

$\frac{CE}{EA} \bullet \frac{2}{1} \bullet \frac{1}{3} = 1$ ***A1***

$\frac{CE}{EA} = \frac{3}{2}$ ***AG***

[2 marks]

A.b. (i) using Ceva's theorem in ΔABC ,

$\frac{CE}{EA} \bullet \frac{AF}{FB} \bullet \frac{BD}{DC} = 1$ ***M1***

$\frac{3}{2} \bullet \frac{AF}{FB} \bullet \frac{1}{2} = 1$ ***A1***

$\frac{AF}{FB} = \frac{4}{3}$ ***A1***

(ii) using Menelaus' theorem in ΔABE , with traversal (FC), ***M1***

$\frac{AF}{FB} \bullet \frac{BG}{GE} \bullet \frac{EC}{CA} = -1$ ***A1***

$\frac{4}{3} \bullet \frac{BG}{GE} \bullet \frac{3}{5} = 1$ ***A1***

$\frac{BG}{GE} = \frac{5}{4}$ ***A1***

[7 marks]

B. using Ptolemy's theorem in PAEC, ***M1***

$PA \bullet EC + AE \bullet PC = PE \bullet AC$ ***A1***

since $EC = AE = AC$, ***M1***

$PE = PA + PC$ ***A1***

similarly for PBDF, ***M1***

$PB \bullet DF + BD \bullet PF = PD \bullet BF$ ***(A1)***

$PD = PB + PF$ ***A1***

adding these results,

$PE + PD = PA + PB + PC + PF$ ***AG***

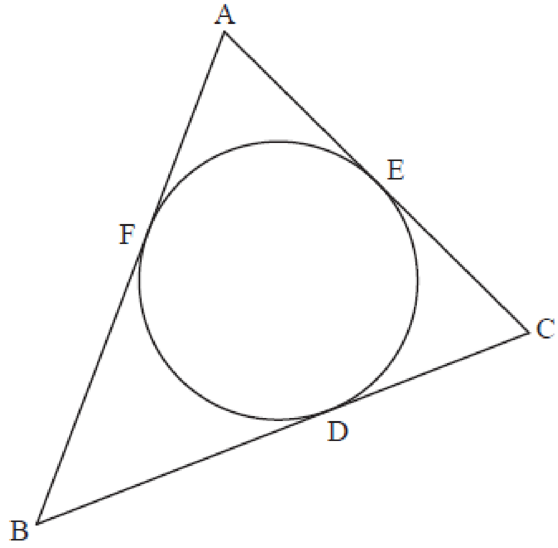
[7 marks]

Examiners report

A.a Part A was reasonably well done by many candidates. It would appear from the scripts that, in general, candidates find Menelaus' Theorem more difficult to apply than Ceva's Theorem, probably because the choice of transversal is not always obvious.

A.b Part A was reasonably well done by many candidates. It would appear from the scripts that, in general, candidates find Menelaus' Theorem more difficult to apply than Ceva's Theorem, probably because the choice of transversal is not always obvious.

B. Few fully correct answers were seen to part B with most candidates unable to identify which cyclic quadrilaterals should be used.



a. The diagram shows triangle ABC together with its inscribed circle. Show that [AD], [BE] and [CF] are concurrent. [8]

b. PQRS is a parallelogram and T is a point inside the parallelogram such that the sum of $\mathrm{P\hat{T}Q}$ and $\mathrm{R\hat{T}S}$ is 180° . Show that [13]

$\mathrm{TP} \times \mathrm{TR} + \mathrm{ST} \times \mathrm{TQ} = \mathrm{PQ} \times \mathrm{QR}$.

Markscheme

a. Since the lengths of the two tangents from a point to a circle are equal ***(M1)***

$\mathrm{AF} = \mathrm{AE}, \mathrm{BF} = \mathrm{BD}, \mathrm{CD} = \mathrm{CE}$ ***A1***

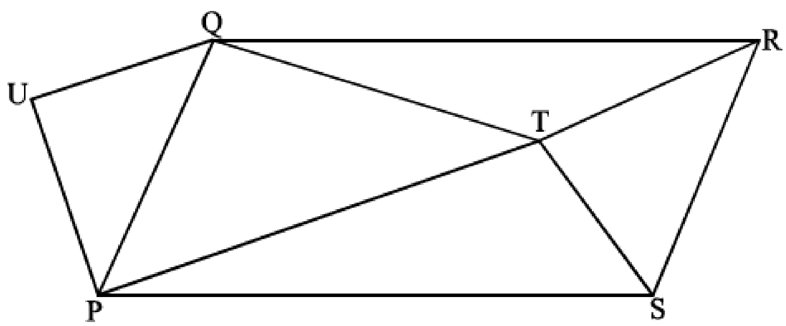
Consider

$\frac{\mathrm{AF}}{\mathrm{FB}} \times \frac{\mathrm{BD}}{\mathrm{DC}} \times \frac{\mathrm{CE}}{\mathrm{EA}} = 1$ (signed lengths are not relevant here) ***M2A2***

It follows by the converse to Ceva's Theorem that [AD], [BE], [CF] are concurrent. ***R2***

[8 marks]

b.



Draw the $\Delta\mathrm{PQU}$ congruent to $\Delta\mathrm{SRT}$ (or translate $\Delta\mathrm{SRT}$ to form $\Delta\mathrm{PQU}$). ***R2***

Since $\mathrm{P\hat{U}Q} = \mathrm{S\hat{T}R}$, and $\mathrm{S\hat{T}R} + \mathrm{P\hat{T}Q} = 180^\circ$ it follows that ***R2***

$\mathrm{P\hat{U}Q} + \mathrm{P\hat{T}Q} = 180^\circ$ ***R1***

The quadrilateral PUQT is therefore cyclic ***R2***

Using Ptolemy's Theorem, ***M2***

$\mathrm{UQ} \times \mathrm{PT} + \mathrm{PU} \times \mathrm{QT} = \mathrm{PQ} \times \mathrm{UT}$ ***A2***

Since $\mathrm{UQ} = \mathrm{TR}$, $\mathrm{PU} = \mathrm{ST}$ and $\mathrm{UT} = \mathrm{QR}$ ***R2***

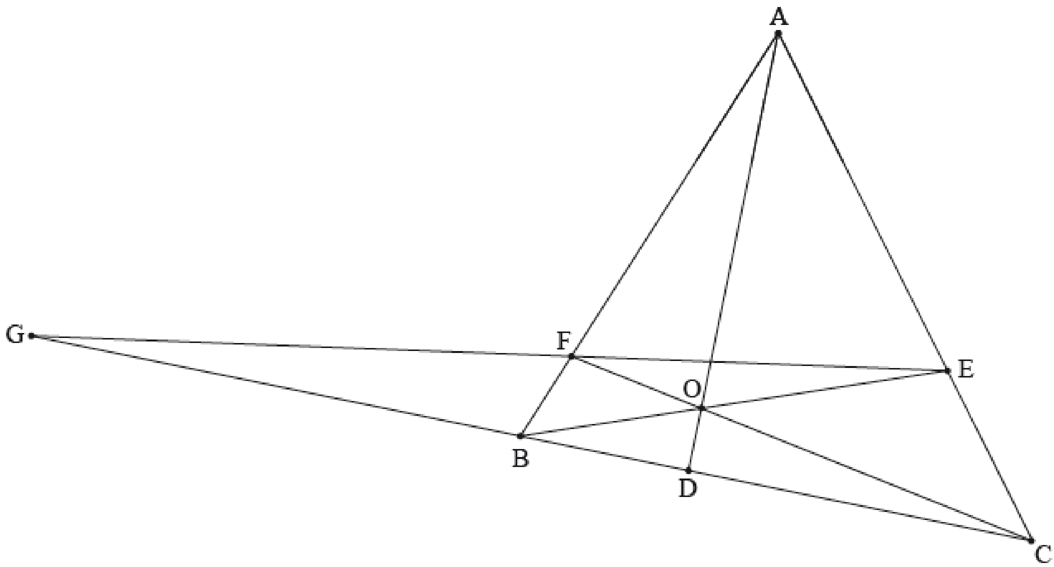
Then $\mathrm{TP} \times \mathrm{TR} + \mathrm{ST} \times \mathrm{TQ} = \mathrm{PQ} \times \mathrm{QR}$ ***AG***

[13 marks]

Examiners report

a. [N/A]

b. [N/A]



The diagram above shows a point O inside a triangle ABC. The lines (AO), (BO), (CO) meet the lines (BC), (CA), (AB) at the points D, E, F respectively. The lines (EF), (BC) meet at the point G.

(a) Show that, with the usual convention for the signs of lengths in a triangle, $\frac{\mathrm{BD}}{\mathrm{DC}} = -\frac{\mathrm{BG}}{\mathrm{GC}}$.

(b) The lines (FD), (CA) meet at the point H and the lines (DE), (AB) meet at the point I. Show that the points G, H, I are collinear.

Markscheme

(a) applying Ceva's theorem to triangle ABC,

$\frac{\mathrm{CD}}{\mathrm{DB}} \times \frac{\mathrm{AE}}{\mathrm{EC}} \times \frac{\mathrm{BF}}{\mathrm{FA}} = 1$ ***M1A1***

applying Menelaus' theorem to triangle ABC with transversal (GFE),

$\frac{\mathrm{BG}}{\mathrm{GC}} \times \frac{\mathrm{CE}}{\mathrm{EA}} \times \frac{\mathrm{AF}}{\mathrm{FB}} - = 1$ ***M1A1***

multiplying the two equations, ***M1***

$\frac{\mathrm{CD}}{\mathrm{DB}} \times \frac{\mathrm{BG}}{\mathrm{GC}} = -1$ ***A1***

so that $\frac{\mathrm{BD}}{\mathrm{DC}} = -\frac{\mathrm{BG}}{\mathrm{GC}}$ ***AG***

[6 marks]

(b) similarly

$\frac{\mathrm{CE}}{\mathrm{EA}} = -\frac{\mathrm{CH}}{\mathrm{HA}}$ ***M1A1***

and $\frac{\mathrm{AF}}{\mathrm{FB}} = -\frac{\mathrm{AI}}{\mathrm{IB}}$ ***A1***

multiplying the three results,

$\frac{\mathrm{BD}}{\mathrm{DC}} \times \frac{\mathrm{CE}}{\mathrm{EA}} \times \frac{\mathrm{AF}}{\mathrm{FB}} = -\frac{\mathrm{BG}}{\mathrm{GC}} \times \frac{\mathrm{CH}}{\mathrm{HA}} \times \frac{\mathrm{AI}}{\mathrm{IB}}$ ***M1A1***

by Ceva's theorem, as shown previously, the left hand side is equal to 1, therefore so is the right hand side ***R1***

that is $\frac{\mathrm{BG}}{\mathrm{GC}} \times \frac{\mathrm{CH}}{\mathrm{HA}} \times \frac{\mathrm{AI}}{\mathrm{IB}} = -1$ ***A1***

it follows from the converse to Menelaus' theorem that G, H, I are collinear ***R1***

[8 marks]

Examiners report

[N/A]

A. Prove that the interior bisectors of two of the angles of a non-isosceles triangle and the exterior bisector of the third angle, meet the sides of the triangle in three collinear points. [8]

B.a An equilateral triangle QRT is inscribed in a circle. If S is any point on the arc QR of the circle, [10]

(i) prove that $\text{ST} = \text{SQ} + \text{SR}$;

(ii) show that triangle RST is similar to triangle PSQ where P is the intersection of [TS] and [QR];

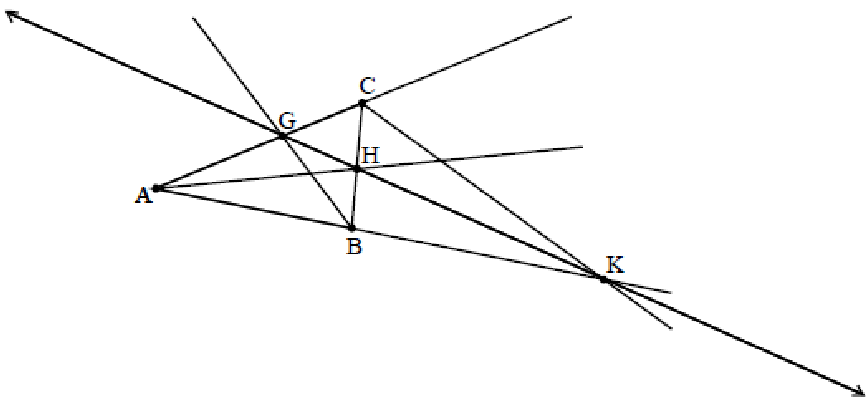
(iii) using your results from parts (i) and (ii) deduce that $\frac{1}{\text{SP}} = \frac{1}{\text{SQ}} + \frac{1}{\text{SR}}$.

B.b Perpendiculars are drawn from a point P on the circumcircle of triangle LMN to the three sides. The perpendiculars meet the sides [LM], [MN] and [LN] at the points E, F and G respectively. [8]

Prove that $\text{PL} \times \text{PF} = \text{PM} \times \text{PG}$.

Markscheme

A.



triangle ABC has interior angle bisectors AH, BG and exterior angle bisector CK

using the angle bisector theorem, ***M1***

$\frac{\text{CH}}{\text{HB}} = \frac{\text{CA}}{\text{AB}}, \frac{\text{AG}}{\text{GC}} = \frac{\text{AB}}{\text{CB}}, \frac{\text{BK}}{\text{AK}} = \frac{\text{CB}}{\text{CA}}$ ***A2***

hence, $\frac{\text{CH}}{\text{HB}} \times \frac{\text{AG}}{\text{GC}} \times \frac{\text{BK}}{\text{AK}} = \frac{\text{CA}}{\text{AB}} \times \frac{\text{AB}}{\text{CB}} \times \frac{\text{CB}}{\text{CA}} = 1$ ***M1A1***

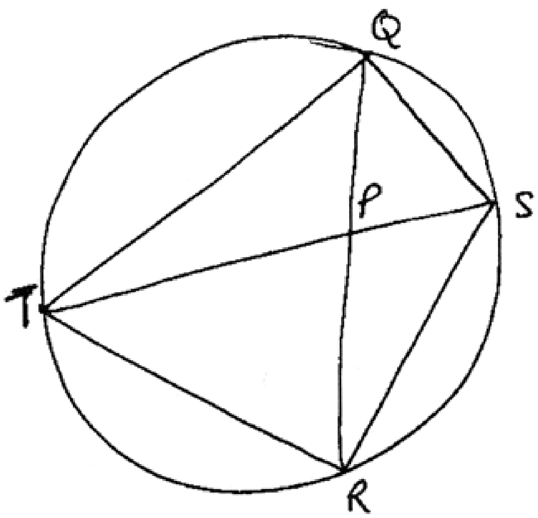
but $\frac{\text{BK}}{\text{AK}} = -\frac{\text{BK}}{\text{KA}}$ ***R1***

so, $\frac{\text{CH}}{\text{HB}} \times \frac{\text{AG}}{\text{GC}} \times \frac{\text{BK}}{\text{AK}} = -1$ ***A1***

hence, by converse Menelaus' theorem, G, H and K are collinear ***R1***

[8 marks]

B.a(i)



$ST \bullet QR = SQ \bullet RT + SR \bullet QT$ ***M1A1***

but $QT = QR = RT$ ***R1***

hence $ST = SQ + SR$ ***AG***

(ii) $\angle STR = \angle SQR$ ***R1***

$\angle QST = \angle QRT = 60^\circ$ ***A1***

$\angle RST = \angle RQT = 60^\circ$ ***A1***

hence $\angle QST = \angle RST$ and $\Delta RST \sim \Delta PSQ$ ***R1***

(iii) $\frac{ST}{SQ} = \frac{SR}{SP} \rightarrow ST \times SP = SR \times SQ$ ***M1A1***

but $ST = (SQ + SR)$

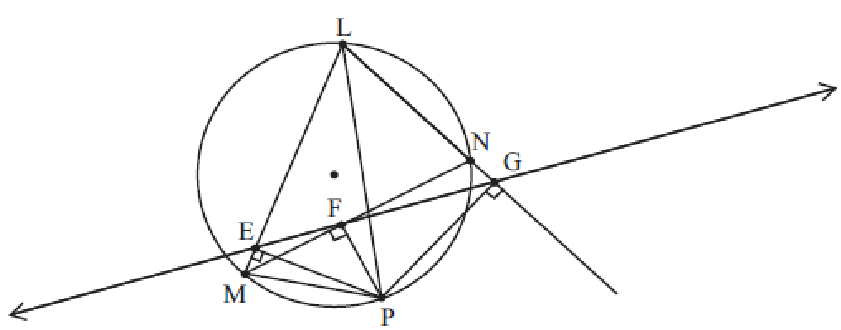
$(SQ + SR) \times SP = SR \times SQ$ ***A1***

$SQ \times SP + SR \times SP = SR \times SQ$

hence $\frac{1}{SP} = \frac{1}{SQ} + \frac{1}{SR}$ ***AG***

[10 marks]

B.b.



sine $m\angle PEM \cong m\angle PFM \cong 90^\circ$

then quadrilateral PFEM is cyclic ***M1A1***

so $m\angle PME \cong m\angle PFG$

since $m\angle PGL \cong m\angle PEL \cong 90^\circ$

then quadrilateral PGLE is cyclic

so $m\angle\text{PGE} \cong m\angle\text{PLE}$ ***R1A1***

now E, F and G are collinear since they are on the Simson line of ΔLMN ***R1***

hence $\Delta\text{PFG} \sim \Delta\text{PML}$ ***A1***

so $\frac{\text{PL}}{\text{PM}} = \frac{\text{PG}}{\text{PF}} \Rightarrow \text{PL} \times \text{PF} = \text{PM} \times \text{PG}$ ***R1AG***

[8 marks]

Examiners report

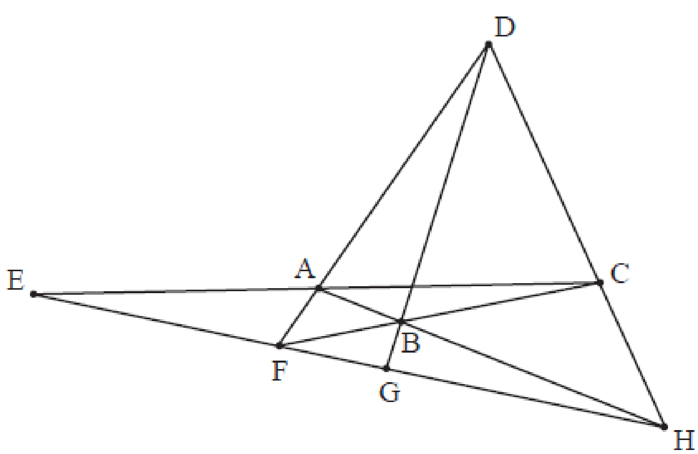
A. As in paper 1 there is a sad lack of knowledge of geometry although some good solutions were seen and at least one school is using techniques very successfully that are not mentioned in the program.

B.a. Part (a) was well done.

B.b. Few clear solutions to part (b) were seen.

ABCD is a quadrilateral. (AD) and (BC) intersect at F and (AB) and (CD) intersect at H. (DB) and (CA) intersect (FH) at G and E respectively.

This is shown in the diagram below.



Prove that $\frac{\text{HG}}{\text{GF}} = -\frac{\text{HE}}{\text{EF}}$.

Markscheme

in ΔHFD , [HA], [FC] and [DG] are concurrent at B ***M1***

so, $\frac{\text{HG}}{\text{GF}} \times \frac{\text{FA}}{\text{AD}} \times \frac{\text{DC}}{\text{CH}} = 1$ by Ceva's theorem ***A1R1***

in ΔHFD , with CAE as transversal, ***M1***

$\frac{\text{HE}}{\text{EF}} \times \frac{\text{FA}}{\text{AD}} \times \frac{\text{DC}}{\text{CH}} = -1$ by Menelaus' theorem ***A1R1***

therefore, $\frac{\text{HG}}{\text{GF}} = -\frac{\text{HE}}{\text{EF}}$ ***AG***

[6 marks]

Examiners report

This was not a difficult question but again too many candidates often left gaps in their solutions perhaps thinking that what they were doing was obvious and needed no written support

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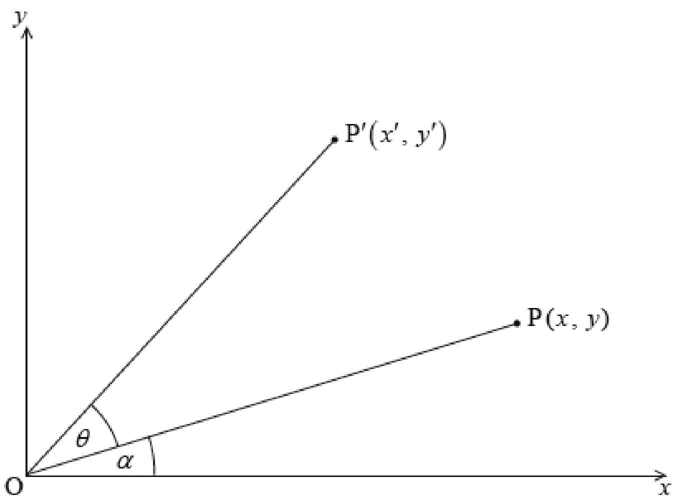


The diagram above shows the points $\mathrm{P}(x,\ y)$ and $\mathrm{P}'(x',\ y')$ which are equidistant from the origin O. The line (OP) is inclined at an angle α to the x-axis and $\mathrm{P\hat{O}P'} = \theta$.

(a) (i) By first noting that $\mathrm{OP} = x\sec\alpha$, show that $x' = x\cos\theta - y\sin\theta$ and find a similar expression for y'.

(ii) Hence write down the 2×2 matrix which represents the anticlockwise rotation about O which takes P to P'.

(b) The ellipse E has equation $5x^2 + 5y^2 - 6xy = 8$.

(i) Show that if E is rotated **clockwise** about the origin through 45°, its equation becomes $\frac{x^2}{4} + y^2 = 1$.

(ii) Hence determine the coordinates of the foci of E.

Markscheme

(a) (i) $x' = x\sec\alpha\cos(\theta + \alpha)$ ***M1***

$= x\sec\alpha(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$ ***A1***

$= x\cos\theta - x\tan\alpha\sin\theta$ ***A1***

$= x\cos\theta - y\sin\theta$ ***AG***

$y' = x\sec\alpha\sin(\theta + \alpha)$ ***M1***

$= x\sec\alpha(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$ ***A1***

$= x\sin\theta + x\tan\alpha\cos\theta$

$= x\sin\theta + y\cos\theta$ ***A1***

(ii) the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ represents the rotation ***A1***

[7 marks]

(b) (i) the above relationship can be written in the form

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$ ***M1***

let $\theta = -\frac{\pi}{4}$

$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}$ ***A1***

$y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$

substituting in the equation of the ellipse,

$5\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)^2 + 5\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right)^2 - 6\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) = 8$ ***M1***

$5\left(\frac{x'^2}{2} + \frac{y'^2}{2} - x'y'\right) + 5\left(\frac{x'^2}{2} + \frac{y'^2}{2} + x'y'\right) - 6\left(\frac{x'^2}{2} - \frac{y'^2}{2}\right) = 8$ ***A1***

leading to $\frac{x'^2}{4} + y'^2 = 1$ ***AG***

Note: Award ***M1A0M1A0*** for using $\theta = \frac{\pi}{4}$ leading to $\frac{y'^2}{4} + x'^2 = 1$.

(ii) in the usual notation, $a = 2, b = 1$ ***(M1)***

the coordinates of the foci of the rotated ellipse are $\left(\sqrt{3}, 0\right)$ and $\left(-\sqrt{3}, 0\right)$ ***A1***

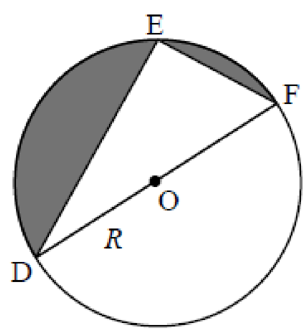
the coordinates of the foci of E are therefore $\left(\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\right)$ and $\left(\frac{-\sqrt{3}}{\sqrt{2}}, \frac{-\sqrt{3}}{\sqrt{2}}\right)$ ***A1***

[7 marks]

Examiners report

[N/A]

A new triangle DEF is positioned within a circle radius *R* such that DF is a diameter as shown in the following diagram.



a.i. In a triangle ABC, prove $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. [4]

a.ii. Prove that the area of the triangle ABC is $\frac{1}{2}ab\sin \mathrm{C}$. [2]

a.iii. Given that *R* denotes the radius of the circumscribed circle prove that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$. [2]

a.iv. Hence show that the area of the triangle ABC is $\frac{abc}{4R}$. [2]

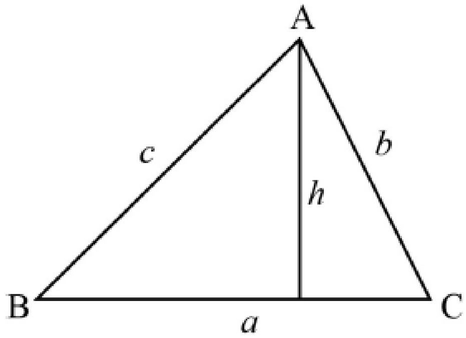
b.i. Find in terms of *R*, the two values of $(\mathrm{DE})^2$ such that the area of the shaded region is twice the area of the triangle DEF. [9]

b.ii. Using two diagrams, explain why there are two values of $(\mathrm{DE})^2$. [2]

c. A parallelogram is positioned inside a circle such that all four vertices lie on the circle. Prove that it is a rectangle. [3]

Markscheme

a.i.



$\sin \mathrm{B} = \frac{h}{c}$ and $\sin \mathrm{C} = \frac{h}{b}$ ***M1A1***

hence $h = c \sin \mathrm{B} = b \sin \mathrm{C}$ ***A1***

by dropping a perpendicular from B, in exactly the same way we find $c \sin \mathrm{A} = a \sin \mathrm{C}$ ***R1***

hence $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

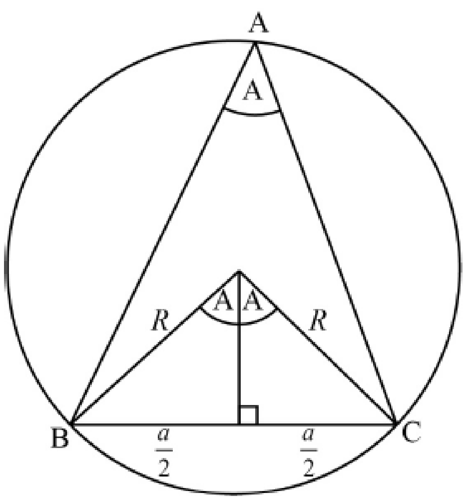
[4 marks]

a.ii. area = $\frac{1}{2}ah$ ***M1A1***

= $\frac{1}{2}ab \sin \mathrm{C}$ ***AG***

[2 marks]

a.iii.



since the angle at the centre of circle is twice the angle at the circumference $\sin A = \frac{a}{2R}$ ***M1A1***

hence $\frac{a}{\sin A} = 2R$ and therefore $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ ***AG***

[2 marks]

a.iv. area of the triangle is $\frac{1}{2}ab \sin \mathrm{C}$ ***M1***

since $\sin \mathrm{C} = \frac{c}{2R}$ ***A1***

area of the triangle is $\frac{1}{2}ab\,\frac{c}{2R} = \frac{abc}{4R}$ ***AG***

[2 marks]

b.i. area of the triangle is $\frac{\pi R^2}{6}$ ***(M1)A1***

$(\mathrm{DE})^2 + (\mathrm{EF})^2 = 4R^2$ ***M1***

$(\mathrm{DE})^2 = 4R^2 - (\mathrm{EF})^2$

$\frac{1}{2}(\mathrm{DE})(\mathrm{EF}) = \frac{\pi R^2}{6} \Rightarrow (\mathrm{EF}) = \frac{\pi R^2}{3(\mathrm{DE})}$ ***M1A1***

$(\mathrm{DE})^2 = 4R^2 - \frac{\pi^2 R^4}{9(\mathrm{DE})^2}$ ***A1***

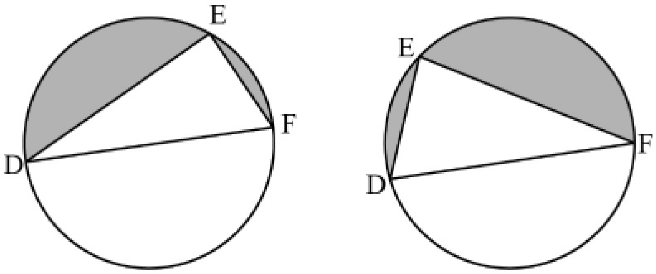
$9(\mathrm{DE})^4 - 36(\mathrm{DE})^2 R^2 + \pi^2 R^4 = 0$ ***A1***

$(\mathrm{DE})^2 = \frac{36R^2 \pm \sqrt{1296R^4 - 36\pi^2 R^4}}{18}$ ***M1***

$(\mathrm{DE})^2 = \frac{36R^2 \pm 6R^2\sqrt{36-\pi^2}}{18} \left(= \frac{6R^2 \pm R^2\sqrt{36-\pi^2}}{3}\right)$ ***A1***

[9 marks]

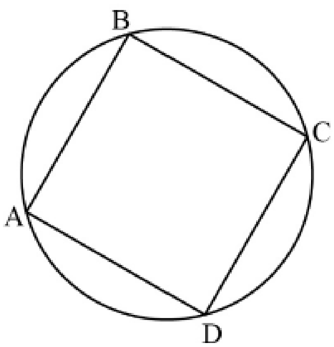
b.ii.



A1A1

[2 marks]

c.



$\hat{\mathrm{A}} + \hat{\mathrm{C}} = 180^\circ$ (cyclic quadrilateral) ***R1***

however $\hat{\mathrm{A}} = \hat{\mathrm{C}}$ (ABCD is a parallelogram) ***R1***

$\hat{\mathrm{A}} = \hat{\mathrm{C}} = 90^\circ$ ***A1***

$\hat{\mathrm{B}} = \hat{\mathrm{D}} = 90^\circ$

hence ABCD is a rectangle ***AG***

[3 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
a.iii. [N/A]
a.iv. [N/A]
b.i. [N/A]
b.ii. [N/A]
c. [N/A]